q-Virasoro constraints and the algebra of Wilson loops in 3d

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Based on:
arxiv:2107.07525 with M.Zabzine
arXiv:2007.10354 with R.Lodin and M.Zabzine
arXiv:1909.10352 with R.Lodin, A.Popolitov and M.Zabzine

Introduction

Supersymmetric localization gives a way to compute partition functions and BPS observables exactly.

Using Coulomb branch localization one has to integrate over the Lie algebra of the gauge group.

The result often takes the form of an integral over random matrices.

Some matrix models are exactly solvable and one can use techniques from integrability and 2d CFTs (Virasoro symmetry, W-algebras, etc.).

This general idea is called **BPS/CFT correspondence** [Nekrasov]

Introduction

The most famous example is AGT, which relates 4d $\mathcal{N} = 2$ to 2d Liouville/Toda CFT [Alday-Gaiotto-Tachikawa][Wyllard]

$$\underbrace{Z^{\text{inst}}}_{\text{Nekrasov instanton function}} = \underbrace{\langle G|G\rangle}_{\text{Gaiotto-Whittaker vector}}$$

There is a 5d version that relates $\mathcal{N} = 1$ gauge theories to q-Liouville correlators [Awata-Yamada]

q is a deformation parameter related to the geometry of the background.

Introduction

Today we are interested in the 3d version of BPS/CFT correspondence:

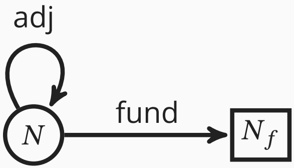
- On the gauge theory side we have 3d $\mathcal{N} = 2$ supersymmetric gauge theory
- On the q-CFT side we have an highest weight module for q-Virasoro

Plan of the talk

- 3d $\mathcal{N} = 2$ partition functions as matrix integrals
- q-Virasoro constraints and their solution
- Averages of characters (Macdonald functions)
- Refined Chern-Simons and refined ABJ

3d Gauge theory

We consider $U(N)$ Yang-Mills-Chern-Simons theory on $D^2 \times_q S^1$



with CS level κ_{CS} and FI parameter ξ_{FI}.

$D^2 \times_q S^1$ partition function

Upon localization we have the matrix integral [Beem-Dimofte-Pasquetti]

$$Z = \oint \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{x_i} \mathrm{e}^{S_{\text{classic}}(x_i)} \underbrace{\prod_{i \neq j} \frac{(x_i/x_j; q)_\infty}{(t x_i/x_j; q)_\infty}}_{\text{vector+adjoint}} \underbrace{\prod_{i=1}^{N} \prod_{a=1}^{N_f} (q x_i u_a; q)_\infty}_{\text{fundamentals}}$$

with:

- x_i = gauge variables ($i = 1, \ldots, N$)
- u_a = fundamental masses ($a = 1, \ldots, N_f$)
- t = adjoint mass
- q = holonomy of D^2 over S^1, $\quad (x;q)_\infty := \prod_{n=0}^{\infty}(1 - q^n x)$

1/2-BPS **Wilson loop** operators are computed as insertions of characters

$$\langle \mathrm{Schur}_\lambda(x_i) \rangle = \oint \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{x_i} \, \mathrm{Schur}_\lambda(x_i) \cdot [\ldots]$$

Generating function of Wilson loops

From the matrix model point-of-view it is convenient to collect all Wilson loop operators into a generating function of times $\tau = \{\tau_1, \tau_2, \dots\}$

$$Z(\tau) := \left\langle \exp\left(\sum_{k=1}^{\infty} \frac{\tau_k p_k}{k}\right)\right\rangle \overset{\text{Cauchy}}{=} \sum_{\lambda} \underbrace{\langle \mathrm{Schur}_\lambda(p)\rangle}_{\textbf{Wilson loop}} \cdot \mathrm{Schur}_\lambda(\tau), \qquad p_k := \sum_{i=1}^{N} x_i^k$$

The **Schur functions** form a basis of the ring of symmetric functions in x_i, hence we can use them to expand any other symmetric function.

$$f(x_i) = \sum_{\lambda} f_\lambda \, \mathrm{Schur}_\lambda(x_i)$$

Computing averages of Schurs by explicit integration is hard!
We want an algebraic procedure independent of the rank N.

Ward identities

Relations between correlation functions (i.e. Ward identities) are encoded as differential/difference equations for $Z(\tau)$. Solving these equations can be easier than computing integrals! (N **is just a parameter**)

Ward identities for classical matrix models are obtained by inserting total derivatives in the matrix integral

$$\int \prod_{i=1}^{N} \mathrm{d}x_i \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[x_i^{n+1} (\dots) \right] = 0, \qquad n \geq -1$$

corresponding to the Lie derivative along the vectors $\sum_i x_i^{n+1} \frac{\partial}{\partial x_i}$ which form a Virasoro algebra. These Ward identities are called **Virasoro constraints**.

Example

Consider the integral

$$Z(\tau) = \int \mathrm{d}x \, \mathrm{e}^{-\frac{x^2}{2} + \sum_k \frac{\tau_k x^k}{k}}$$

Insert the total derivative operator $\partial_x[x^{n+1} \dots]$

$$\begin{aligned}\int \mathrm{d}x \, \partial_x \left[x^{n+1} \mathrm{e}^{-\frac{x^2}{2} + \sum_k \frac{\tau_k x^k}{k}} \right] &= (n+1)\langle x^n \rangle - \langle x^{n+2} \rangle + \sum_{k=1}^{\infty} \tau_k \langle x^{n+k} \rangle \\ &= \left[(n+1)n \frac{\partial}{\partial \tau_n} - (n+2) \frac{\partial}{\partial \tau_{n+2}} + \sum_{k=1}^{\infty} \tau_k (n+k) \frac{\partial}{\partial \tau_{n+k}} \right] Z(\tau) = 0\end{aligned}$$

Independent from the contour if there are no boundary contributions!

q-Virasoro constraints

Our matrix model is q-deformed therefore the Ward identities should also be q-deformed. The first guess is to substitute the total derivative by a finite difference operator that depends on q and t [Mironov-Morozov-Zenkevich],

$$\oint \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{x_i} \sum_{i=1}^{N} (T_{q,x_i} - 1) \left[x_i^n \prod_{j \neq i} \frac{1 - t x_i / x_j}{1 - x_i / x_j} (\dots) \right] = 0$$

where $T_{q,x_i} : x_i \mapsto q x_i$ is a shift operator w.r.t. which the q-Pochhammers are quasi-periodic

$$T_{q,x}(x;q)_\infty = \frac{1}{(1-x)} (x;q)_\infty$$

In the limit $q, t \to 1$ we recover the usual derivative.

q-Virasoro constraints

We want to rewrite the constraints as infinitely many **PDEs** for $Z(\tau)$.

This might fail for two reasons:

- the $n = -1$ constraint contains terms proportional to $\sum_i \langle 1/x_i \rangle$. We need to set the coefficient to zero:

$$\Rightarrow \qquad \xi_{\text{FI}} = \frac{\log t}{\log q}(N-1) + 1$$

- the CS action contributes logarithmic terms in x_i. We choose vanishing level

$$\kappa_{\text{CS}} = 0$$

The q-Virasoro module

We can interpret these constraint equations algebraically as the (bosonized) action of q-Virasoro generators on a highest weight state

$$T_n|Z\rangle = 0, \qquad n \geq -1$$

with q-Virasoro current $T(z) = \sum_{n\in\mathbb{Z}} T_n z^n$ realized as a difference operator

$$T(z) = (q/t)^{\frac{1}{2}} \exp\left(-\sum_{k=1}^{\infty} z^{-k} \frac{(1-q^k)}{(1+q^k t^{-k})} \frac{\tau_k}{k}\right) \exp\left(\sum_{k=1}^{\infty} z^k (1-t^{-k}) \frac{\partial}{\partial \tau_k}\right) t^N +$$
$$+ (q/t)^{-\frac{1}{2}} \prod_{a=1}^{N_f} (1 - q u_a/z) \exp\left(\sum_{k=1}^{\infty} z^{-k} \frac{(1-q^k) q^k t^{-k}}{(1+q^k t^{-k})} \frac{\tau_k}{k}\right) \exp\left(\sum_{k=1}^{\infty} z^k \frac{(1-t^k)}{q^k} \frac{\partial}{\partial \tau_k}\right) t^{-N}$$

q-Virasoro algebra

q-Virasoro is an associative deformation of Virasoro inspired by work on quantum groups [Frenkel-Reshetikhin] and quantum integrable models [Shiraishi-Kubo-Awata-Odake].

It is generated by $T_{n\in\mathbb{Z}}$ with relations

$$[T_n, T_m] = -\sum_{l=1}^{\infty} f_l(T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) - \frac{(1-q)(1-t^{-1})}{(1-qt^{-1})}(q^n t^{-n} - q^{-n}t^n)\delta_{n+m,0}$$

and coefficients f_l given by

$$f(z) = \sum_{l=0}^{\infty} f_l z^l = \exp\left(\sum_{k=1}^{\infty} \frac{(1-q^k)(1-t^{-k})}{(1+q^k t^{-k})}\frac{z^k}{k}\right)$$

Solving the constraints

Expanding $Z(\tau)$ on the basis of monomials in $\mathbb{C}[[\tau_1, \tau_2, \dots]]$ we find linear relations between the correlators $\langle p_\lambda \rangle$. These relations are **upper triangular** w.r.t. the some ordering on the partitions λ and we can try to solve them recursively.

For $N_f = 1, 2$ the kernel is **1-dimensional** and the solution is **unique** up to normalization.

For $N_f \geq 3$ the kernel is **∞-dimensional** and more initial data is needed.

Example: For $N_f = 3$ we cannot solve for correlators of the form $\langle p_1^n \rangle$.

Semi-classical limit

How do we recover usual Virasoro?

$$q = \mathrm{e}^{\hbar} \qquad t = q^{\beta} \qquad \hbar \to 0$$

The limit is **ill-defined** if the masses u_a are kept constant.

$$\Rightarrow \quad \sum_a u_a^k = (q^{-k} - 1) g_k, \qquad k = 1, \dots, N_f$$

After re-parametrization of the masses the limit exists and it is an **Hermitian matrix model**.

$$\{q\text{-Virasoro constr. } T_n\} \xrightarrow{\hbar \to 0} \{\text{Virasoro constr. } L_n\}$$

Semi-classical limit

Around $\hbar = 0$ the partition function is

$$Z_{q,t} = \int \prod_{i=1}^{N} \mathrm{d}x_i \prod_{k \neq l} (x_k - x_l)^\beta \prod_{i=1}^{N} \mathrm{e}^{-V(x_i)} + O(\hbar)$$

with **polynomial** potential

$$V(x) = g_1 x + \frac{1}{2} g_2 x^2 + \cdots + \frac{1}{N_f} g_{N_f} x^{N_f}$$

- $N_f = 1$ is Wishart-Laguerre
- $N_f = 2$ is Gaussian
- $N_f \geq 3$ is generalization of Airy function (Dijkgraaf-Vafa phases)

Averages of characters

Schur polynomials are classical characters of $U(N)$. Their averages in the Hermitian matrix model satisfy the **super-integrability** property

$$\langle \text{character} \rangle \sim \text{character}$$

[Morozov-Popolitov-Shakirov] conjectured that in (q,t)-deformed matrix models super-integrability holds for Macdonald polynomials $P_\lambda(x_i;q,t)$.

We propose the following formula for $N_f = 2$ [LC-Lodin-Zabzine]

$$\frac{\langle P_\lambda(x_i)\rangle}{\langle 1\rangle} = \frac{P_\lambda\left(p_k = \frac{(u_1^{-k}+u_2^{-k})}{1-t^k}\right)}{P_\lambda\left(p_k = \frac{1}{1-t^k}\right)} \underbrace{P_\lambda\left(p_k = \frac{1-t^{kN}}{1-t^k}\right)}_{\text{quantum dimension of } R_\lambda}$$

One can plug this formula in the character expansion of $Z(\tau)$ to get a complete solution of the model.

3-Sphere partition function

The 3-sphere decomposes as the union of 2 solid tori

$$S^3 \simeq D^2 \times_{q_1} S^1 \sqcup S^1 \times_{q_2} D^2 \qquad q_1 = \mathrm{e}^{2\pi\mathrm{i}\frac{\omega_2}{\omega_1}},\; q_2 = \mathrm{e}^{2\pi\mathrm{i}\frac{\omega_1}{\omega_2}}$$

The partition function is computed by localization [Hama-Hosomichi-Lee]

$$Z_{S^3} = \int \prod_{i=1}^{N} \mathrm{d}X_i \underbrace{\mathrm{e}^{S_{\text{classic}}(X_i)}}_{\text{CS+FI}} \prod_{i\neq j} \frac{S_2(X_i - X_j|\underline{\omega})}{S_2(X_i - X_j + M|\underline{\omega})} \prod_{i=1}^{N} \prod_{a=1}^{N_f} S_2(\omega_1 + \omega_2 + X_i + m_a|\underline{\omega})$$

where

$$S_2(X_i|\underline{\omega}) \approx (x_{i,1}; q_1)_\infty (x_{i,2}; q_2)_\infty$$

$\omega_{1,2}$ = squashing

M = adjoint mass

m_a = fundamental masses

Modular double

The S^3 generating function factorizes as a product [LC-Lodin-Popolitov-Zabzine]

$$Z_{S^3}(\tau,\tilde{\tau}) \simeq Z_{q_1,t_1}(\tau) \otimes Z_{q_2,t_2}(\tilde{\tau})$$

There are two commuting copies of q-Virasoro, giving the structure of a **modular double** [Nedelin-Nieri-Zabzine].

Analytically more subtle: need to impose $\kappa_{\mathrm{CS}} = N_f/2$ together with **balancing condition** (from T_{-1} constraint) [LC-Lodin-Zabzine]

$$\xi_{\mathrm{FI}} = \omega + M(N-1) - \frac{\omega}{2}\frac{N_f}{2} - \sum_{k=1}^{N_f} \frac{m_k}{2}$$

Refined Chern–Simons

There exist a Macdonald deformation of the modular matrices S and T [Kirillov] which can be used to define a refinement of pure CS partition function [Aganagic-Shakirov]

$$Z^{\mathrm{rCS}} = \langle 0|TST|0\rangle = \int \prod_{i\neq j} \frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty} \prod_{i=1}^{N} \mathrm{e}^{-\frac{\log^2 x_i}{2\log q}} \mathrm{d}x_i$$

$$q = \mathrm{e}^{\frac{2\pi \mathrm{i}}{\kappa+\beta N}} \qquad t = \mathrm{e}^{\frac{2\pi \mathrm{i}\beta}{\kappa+\beta N}}$$

Wilson loops observables are given by Macdonald polynomials

$$W_\lambda(\bigcirc) = \langle R_\lambda|TST|0\rangle = \langle P_\lambda(x_i;q,t)\rangle \qquad (\textbf{unknot superpolynomial})$$

In the $\beta \to 1$ limit we obtain unrefined CS theory

Refined Chern–Simons

Same measure as $D^2 \times S^1$ but different **potential**! Can we find q-Virasoro?

Observation: $$T_{q,x} \mathrm{e}^{-\frac{\log^2 x}{2 \log q}} = \left(q^{-\frac{1}{2}} x^{-1}\right) \mathrm{e}^{-\frac{\log^2 x}{2 \log q}}$$

We can use the same difference operator to derive Ward identities for the generating function of correlators. The constraints take the form

$$T_n \, Z^{\mathrm{rCS}}(\tau) = 0$$

$Z^{\mathrm{rCS}}(\tau)$ **is an highest weight for q-Virasoro!**

Solution to q-Virasoro constraints

The constraints induce recursion relations on the coefficients of the generating function $Z^{\mathrm{rCS}}(\tau)$. We can solve them explicitly to find the **super-integrability** formula

$$\frac{\langle P_\lambda(x_i)\rangle}{\langle 1\rangle} = \frac{P_\lambda\left(p_k = -\frac{\left(-q^{1/2}\right)^k}{1-t^k}\right)}{P_\lambda\left(p_k = \frac{1}{1-t^k}\right)} P_\lambda\left(p_k = \frac{1-t^{kN}}{1-t^k}\right)$$

which matches an explicit integral evaluation formula by [Cherednik].

Refined ABJ

The matrix model of ABJ theory can also be refined. We can think of it as the **supergroup** $U(N|M)$ version of rCS [Nieri-Pan-Zabzine] [Kimura-Nieri]

$$Z^{\mathrm{rABJ}}_{N,M} = \int \frac{\prod_{i\neq j} \frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty} \prod_{a\neq b} \frac{(y_a/y_b;t^{-1})_\infty}{(q^{-1}y_a/y_b;t^{-1})_\infty}}{\prod_{i,a}(1-\sqrt{t/q}\,x_i/y_a)(1-\sqrt{t/q}\,y_a/x_i)} \prod_{i=1}^{N} \mathrm{e}^{-\frac{\log^2 x_i}{2\log q}} \mathrm{d}x_i \prod_{a=1}^{M} \mathrm{e}^{-\frac{\log^2 y_a}{2\log t^{-1}}} \mathrm{d}y_a$$

Wilson loops are given by **Super-Macdonald** polynomials [Sergeev-Veselov]

$$W_\lambda(\bigcirc) = \langle SP_\lambda(x_i, y_a; q, t)\rangle$$

The difference operator is a generalization of Macdonald-Ruijsenaars

$$x_i \mapsto q x_i \qquad\qquad y_a \mapsto t^{-1} y_a$$

The constraints take the same form as rCS for $N_{\mathrm{eff}} = N - \frac{\log q}{\log t}M$!

Quantum q-geometric Langlands

The q-Virasoro algebra has an outer automorphism that acts on the parameters as

$$q \leftrightarrow t^{-1} \qquad \beta \mapsto \frac{1}{\beta} \qquad N \mapsto -\beta N$$

The normalized generating function of rCS is invariant under Langlands.

For physical values of parameters $q = \mathrm{e}^{\frac{2\pi \mathrm{i}}{\kappa+\beta N}}$ and $t = \mathrm{e}^{\frac{2\pi \mathrm{i}\beta}{\kappa+\beta N}}$ (root of unity)

$$q^{\kappa} t^{N} = 1$$

Langlands duality reduces to 3d Seiberg duality or equivalently level/rank

$$U(N)_{\kappa} \leftrightarrow U(k)_{N}$$

$$W_{\lambda}(\bigcirc) \leftrightarrow W_{\lambda'}(\bigcirc)$$

Summary and Outlook

- Complete solution of q-Virasoro for $N_f = 1, 2$ on $D^2 \times S^1$ and S^3. (i.e. exact computation of all Wilson loops at finite N)
- Semi-classical expansion around Hermitian matrix model (conformal limit).
- Solution of rCS and rABJ matrix models.

To do:

- Beyond the unknot. Toric knots invariants ?
- Use q-Virasoro to solve 5d Nekrasov [Kimura-Pestun] [Nieri-Pan-Zabzine]
- Elliptic Virasoro constraints and elliptic CS [Nieri] [vanDiejen-Görbe]
- Cohomological limit, ϵ-Virasoro and 2d $\mathcal{N} = (2, 2)$ [Nieri-Zenkevich]